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Mathematical Principles for Setting Signal Processing Systems and Regularization

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Abstract—An apparatus function (AF) is selected using the mathematical principles of settings (MPS) so that the norm of its resolving or inverse function would be minimal within limits of tolerance that are less than the AF setting error. This would allow us to obtain solutions to inverse problems with the highest possible accuracy and limiting high resolution. In MPS, the result is reversible and there is no a priori smoothness of the solution with residual error, as there is in regularization.

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INTRODUCTION

Apparatus functions (AFs) arise as a result of calculations or measurements. This means that we can have AFs with a certain accuracy and tolerance, or have an AF set in a defined neighborhood. However, it is known that an AF set is characterized by irregularity in the sense that both reversible and irreversible AFs can be present in a small AF neighborhood. We shall say that an AF set is a regular set if it consists only of AFs that are reversible. In this work, we develop mathematical principles of settings (MPS) that reflect nonregular AF sets in regular sets of reversible and irreversible AFs of two-dimensional MPS grid of parameters *NxDIAP*, where N is the set of lengths of definitional domains and *DIAP* is the limitation in the frequency domain. In regularization [1], we consider the AF to remain unchanged. The poor conditionality of an AF is compensated for by the strange requirement of another object, the a priori smoothness of solutions.

REVERSIBILITY INDICATOR

In optics, it is normal to correlate modulation transfer function (MTF) $M = M(O)$ with AF O. It is obtained as the solution to a proper value problem [2–6] on a set of functions: basis $H = e_{i,}, i = 1 : N\}$, consisting of discrete Fourier harmonics e_i specified in the definitional domain of AF O ($O * H = MH$). Convolution operation * for AF O is thus reduced to multiplying the Fourier harmonics by MTF values $M = M(O)$.

It is known that the orthonormalization of discrete Fourier harmonics $e_i \in H$, $d_{ij} = (e_i, e_j)$, $i, j = 1 : N$ can be done on state-of-the-art computers with accuracy of 10^{-14}–10^{-15}. We therefore associate the less precise instrumental error or interval $Iz = \text{abs}(.) = 10^{-13}$ with the problem of $M(O)$. If $M(O) \in Iz$ for some components, we shall consider that there are no corresponding spectral components of MTF $M(O)$. We naturally calculate complete numerical inversion $M(zR) = 1/M(O)$ if all components $M(O) \notin Iz$. Under these conditions, the problem of $M(O)$ determines two functions: (i) the spectrum of operation or MTF $M = M(O)$ and (ii) inverse function $O = M^{-1}(M(O))$. As with MTF $M(O) \notin Iz$, we have resolving function $zR = M^{-1}(M(zR))$.

The AF O that has a definitional domain with length $N = 25$ points is irreversible. Two MTF values (see Fig. 1) fall into Iz. The resolved AF $R * O(0)$ at zero therefore has a value lower than unity, since two summands are missing from the corresponding incomplete Fourier series for the $R * O$ of cosines.

This prevents us from inverting AF O distortions or compensating for AF distortions in, e.g., image I_x obtained with the linear image-forming model $I_y = O * I_x$, where I_x is the signal or image at input O of the device and I_y is the signal or image at the output. If we apply resolving function zR to images I_y at $N = 25$ points, we obtain on resolved images $zR * I_y$ a moir or grid with lost spatial frequencies, on which $M(O) \in Iz$. The moir or grid is also seen on the resolved AF O or in $zR * O$ (see Fig. 1). Adding one point to the definitional domain of AF O eliminates the two MTF values from instrumental error Iz and AF O is inverted. Depending on lengths N of the definitional domains, the values at zero of function $zR * O(0)$ give us the function that we shall use as a reversibility indicator. We can see from reversibility indicator $II(zR * O)$ in definitional domains with lengths $N = 18 : 32$ that irreversibility is observed three times (see Fig. 1). The MPS is converted into the regular reversible AF set pO with an accuracy of $Err(pO)$. Prefix p of function O means that function pO was obtained as a result of MPS or preliminary preparation. Regular AFs $pR =$

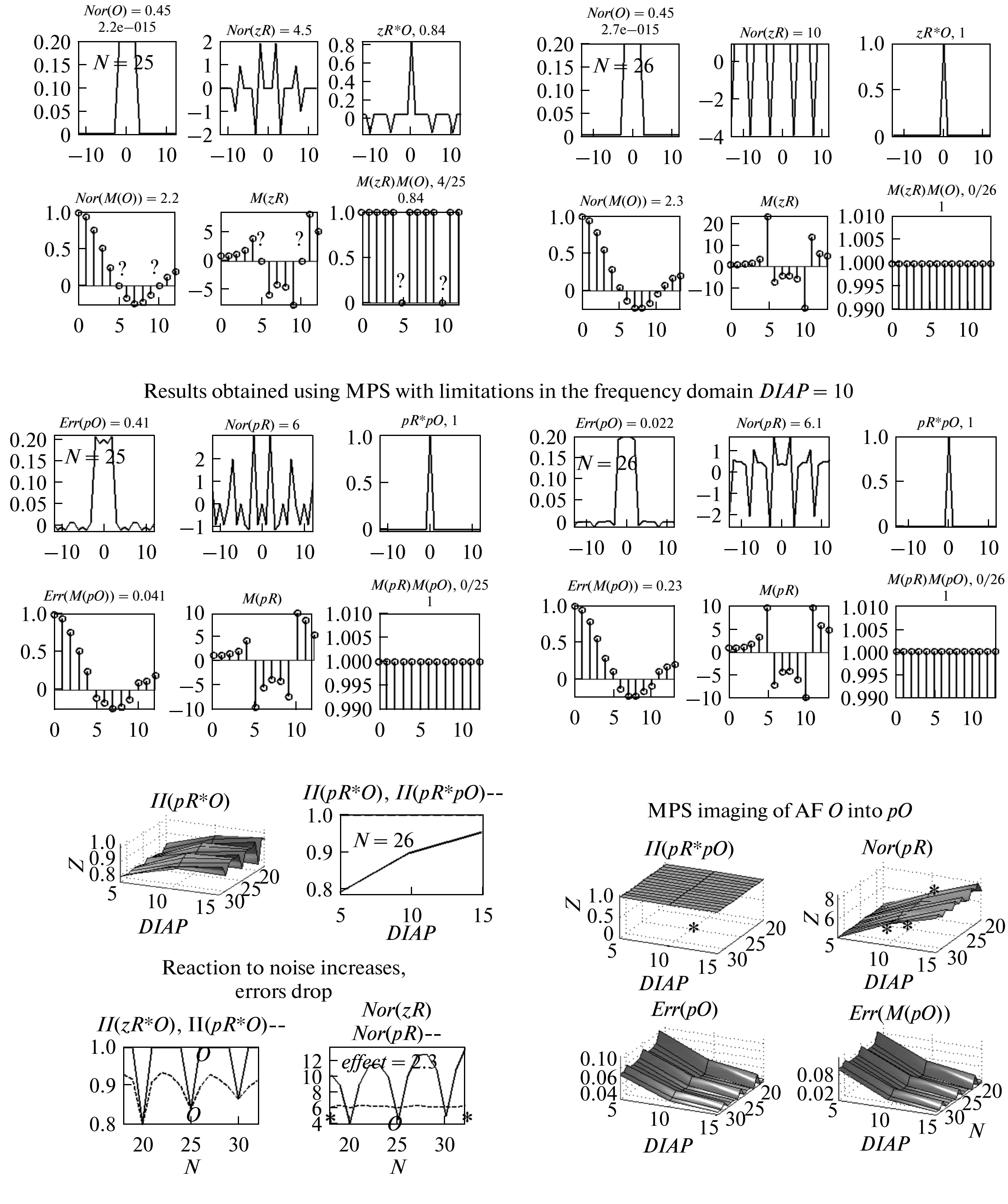

Fig. 1. MPS imaging of AF O into regular pO with reaction to noise *Nor(pR)* and corresponding errors *Err*(pO) and *Err*(pM).

pO^{-1} are specified on grid $PAR = NxDIAP$. With reversibility, the resolved images $zR * I_y = zR * O * I_x = I_x$ at $N = 26$ in our example; i.e., we have complete compensation for AF O. What can prevent complete compensation for AF O distortions?

Most researchers analyze the linear image-forming model with added noise $I_y = O * I_x + \nu$ and $R * I_y = I_x + R * \nu$. If the ν is white normal noise with standard deviation $SD(\nu) = \sigma$, then $SD(R * \nu) = Nor(R)\sigma$. In this sense, the norm of the resolving function determines the reaction to the added noise or the accuracy of the solution to the problem.

MPS TOOLS

We obtain our MPS tools using the *ToolsMPS* function

$$\begin{aligned}\{II, MM, Fun, Nor, Err\} &= ToolsMPS(O, Iz, PAR),\\ II &= [II(zR*O),\ II(pR*pO),\\ &\quad II(pR*O),\ II(R*O)],\\ MM &= [M(zR)M(O),\ M(pR)M(pO),\\ &\quad M(pR)M(O),\ M(R)M(O)],\\ Fun &= [O,\ zR*O,\dots,M(zO)],\\ Nor &= [Nor(O),\dots,Nor(M(zO))],\\ Err &= [Err(zO),\ Err(M(zO)),\ Err(pO),\ Err(M(pO))].\end{aligned} \quad (1)$$

AF O, instrumental error Iz, and grid PAR are at the input of ToolsMPS function (1), and five blocks of tools (II, reversibility indicators; MM, accuracy indicators; FUN, functions and their norms Nor; and error block Err) are at the output. The first block of indicators II contains reversibility indicator $II(zR * O)$, regularity indicator $II(pR * pO)$, mismatch indicator $II(pR * O)$, and brief calculation accuracy indicator $II(R * O)$. The second block of indicators MM contains bandwidth use indicator $M(zR)M(O)$, full bandwidth indicator $M(pR)M(pO)$, efficiency of bandwidth use indicator $M(pR)M(O)$, and long calculation accuracy indicator $M(R)M(O)$. Third block FUN contains the MPS functions. In the fourth block, Nor denotes the norms of functions. Fifth and final block Err represents errors.

Regularization Theorem

On the PAR grid, irregular set AF O is reflected into regular sets with $pO = pR^{-1}$, $II(pR * pO) = 1$.

Monotonicity Theorem

Reaction to noise $Nor(pR)$ and mismatch $II(pR * O)$ increase monotonically while errors $Err(pO)$ and $Err(pM)$ decrease monotonically, since $DIAP$ grows and N is fixed on the PAR grid.

MPS $effect = \|zR\|/\|pR\| \geq 1$ of the reaction reduction to the noise is observed.

Main Problem of MPS

Regular function $pR = pO^{-1}$ is selected by solving the main problem of MPS:

$$\min_{PAR}\{\|pR\| : \|pO - O\| \leq \lim err\},\quad \lim err \leq err. \quad (2)$$

Problem (2) ensures maximum accuracy of the result [2, 6, 7] on the PAR grid by using the instruments of *ToolsMPS* (1).

Effectiveness of Using the Bandwidth of the Signal Processing System

Let us evaluate the effectiveness of using the bandwidth of the signal processing system according to $\underline{M(R)M(O)} = sum(M(R) * M(O))/N$. In the panels below with headings $M(R)M(O)$ and prefixes, this is the lower second parameter. The first upper parameter is the fraction whose numerator contains number of points $M(O)$ that fall into Iz, and whose denominator contains N.

In the panels below with headings $R * O$, we see indicator value $II(R * O)$ with prefixes.

Reversibility Theorems in the Bandwidth of the Signal Processing System

If calculations are performed with instrumental error Iz, equality and inequality ($II(R * O) = \underline{M(R)M(O)} \leq 1$) are maintained with this instrumental error. Inequalities are observed in cases with mismatch error $II(pR * O) < 1$ and residual error of regularization $II(d\text{-}aR * O) = 1\text{-}II(aR * O) < 1$. This means that reversibility is possible only in the part of the bandwidth of the measuring/calculating system that is characterized by $\underline{M(R)M(O)} < 1$.

Reversibility Theorem

Direct reversibility $II(R * O) = 1$, $zR = R = O^{-1}$ follows from Iz reversibility $II(R * O) = 1$ if we use whole bandwidth $\underline{M(zR)M(O)} = 1$ and identity $M(zR)M(O) = 1$.

Controlling Computational Errors

If long calculation accuracy indicator $\underline{M(R)M(O)} \neq 1$ and $M(R)M(O)$ are not equal identical to unity, computational errors accumulate. When this happens, things may not be so bad for brief accuracy indicator $II(R * O) = 1$. The computational process is not controlled if brief indicator $II(R * O) \neq 1$.

COMPARING MPS AND REGULARIZATION

The solution to the variational problem with Tikhonov's functional [1] is reflected in the MPS construction $M(aR) = M(O)/(M(O)^2 + alfa)$, $aR = M^{-1}(M(aR))$. We use prefix a of regularization parameter $alfa$ in the functions. The tools for regularization are also found in the five blocks at the output of the *ToolsREG* function:

$$\begin{aligned}\{II, MM, Fun, Nor, Err\} &= ToolsREG(O, Nalfa)\\ II &= [II(aR*O),\ II(aR*aO)],\\ MM &= [M(aR)M(O),\ M(aR)M(aO)],\\ Fun &= [O, aR*O,\dots,M(aO)],\\ Nor &= [Nor(O),\dots,Nor(M(aO))],\\ Err &= [Err(aO),\ Err(M(aO))]\end{aligned} \quad (3)$$

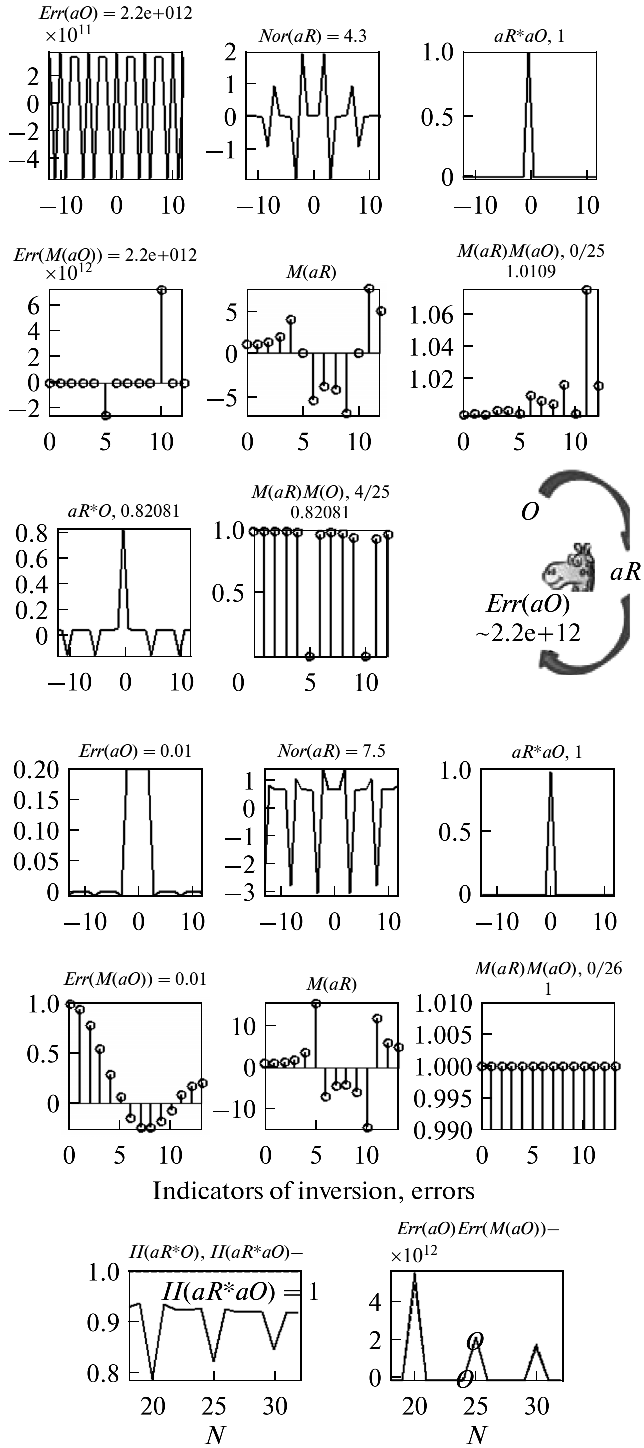

Fig. 2. Results obtained via regularization with parameter *alfa* = 1.e-3, $N = 25$, and $N = 26$.

The new tools for regularization are indicators $II(aR * O)$, $II(aR * aO)$, $M(aR)M(O)$, and $M(aR)M(aO)$, and functions $aR * O$, $aR * aO$ with errors $Err(aO)$ and $Err(M(aO))$.

We construct $M(aO)$, spatial aR, aO from $M(aR)$ for the first time and make sure that in the reversible

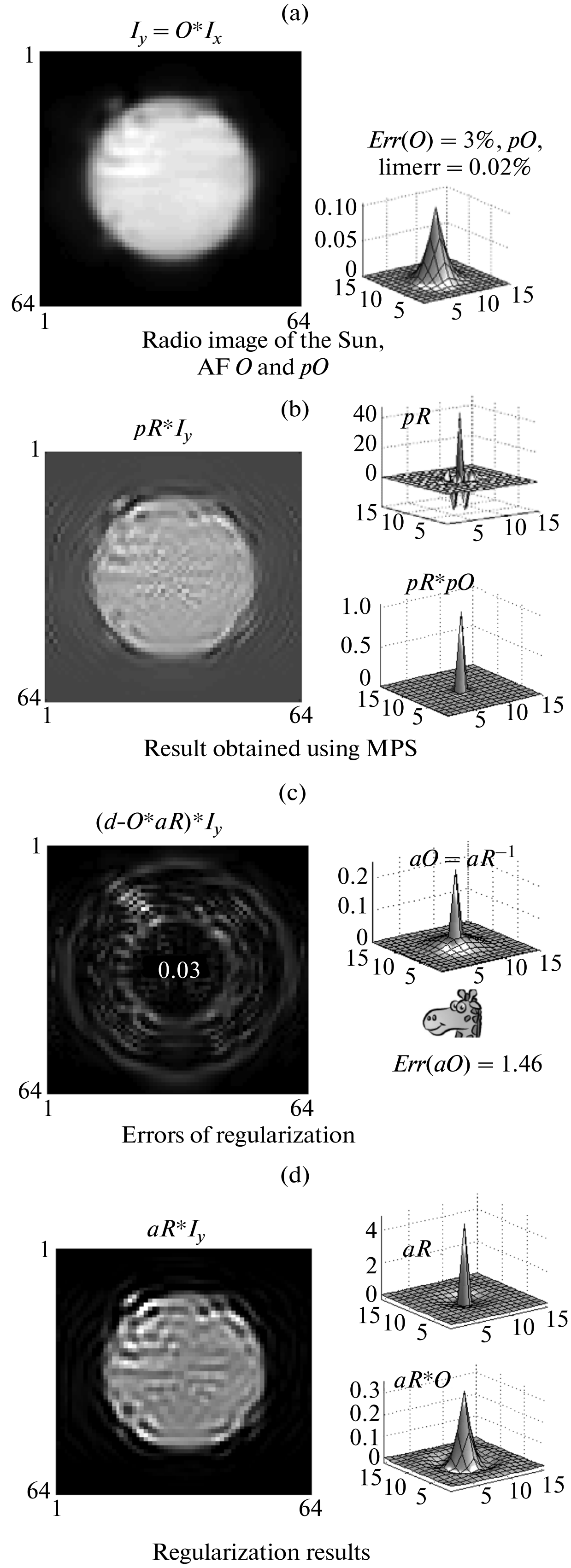

Fig. 3. Comparison of MPS and regularization.

case with $N = 26$, convolution $aR * aO = d$ is the Kronecker symbol and $aR * aO = \underline{M(aR)M(aO)} = 1$ (see Fig. 2). When $N = 25$, values $II(aR * aO) = 1$, $\underline{M(aR)M(aO)} = 1.0109$ indicate a loss of accuracy in long calculations. Regularization and MPS intersect. In the reversible case with $N = 26$, the information on the smoothness is recalculated (as in the MPS) into the deformations in aO and $M(aO)$ with an error of less than 1% (see Figs. 1 and 2). However, the smoothness requirement means that reversibility relative to initial AF O $II(aR * O) < 1$ is lost when $N = 18 : 32$. The Giraffe sign denotes three cases with an enormous error of around $Err(aO) \sim e + 12$, where $II(aR * aO) = 1$; i.e., aR are reversible to aO, which are in turn far from O. We cannot interpret the regularization result with these models of $aR * O$. Let us stress that this cannot happen in MPS, since $pO = pR^{-1}$ and the initial AF O are always close to (2).

MPS FOR RESOLUTION IN ELECTROMAGNETIC IMAGING

A radio-wave image of the Sun in the 3-mm wavelength range is presented below. The AF O (or directional pattern of the antenna) is known with error $Err(O) = 3\%$. We set AF, pO, pR at smaller limits $limerr = 0.02\%$ to achieve the ideal high resolution with maximum accuracy (2).

The AF $aR*O$ with residual error $\max(d – O * aR) * I_y = 3\%$ and $Err(aO) = 1.46$ corresponds to the Sun's regularized image $aR * I_y$ (see the Giraffe sign in Fig. 3). Let us compare the smoothed resolution $aR * I_y$ with the residual error of 3% and the MPS with resolution $pR * I_y$. Our interpretation is that the MPS resolution is reversible ($pR * pO = d$).

CONCLUSIONS

MPS can find practical application in those areas where regularization is now used. It can be also used in more complicated cases, e.g., in problems of focusing laser radiation in a turbulent medium [4], identifying point objects [5], and tuning multi-beam radars [6, 7].

MPS is much more difficult to use than Tikhonov regularization [1]. *ToolsREG* tools (3) were similarly designed for the regularization. We believed that the design should be accompanied by new tools for comparing, establishing, and popularizing the method.

If $\underline{M(R)M(O)} < 1$, the device with the AF O loses information during measurements in the frequency band. We do not believe these losses caused by AF O should be compensated for with the a priori smoothness of solutions. The mathematics of this are perfectly understandable and should explain everything, which is what we did in this work.

In brief, a measuring device with MPS capabilities should be set according to an AF so that in the future, it would be possible to predictably compensate for distortions of regular AFs via ordinary inversion without losses. This is the basic solution to the problem.

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